Math 261-00
2–6–2004

Review Sheet for Test 1

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. (a) Find the vector from $P(2, 3, 0)$ to $Q(5, 5, -4)$.

(b) Find two unit vectors perpendicular to the vector $\langle -12, 5\rangle$.

(c) Find the vector of length 7 that has the same direction as $\langle 1, 2, -2\rangle$.

(d) Find a vector which points in the opposite direction to $\langle 1, -1, 3\rangle$ and has 7 times the length.

(e) Find the scalar component of $\vec{v} = \langle 1, 2, -4\rangle$ in the direction of $\vec{w} = \langle 2, 1, 3\rangle$.

(f) Find the vector projection of $\vec{v} = \langle 1, 1, -4\rangle$ in the direction of $\vec{w} = \langle 2, 0, 5\rangle$.

(g) Find the angle in radians between the vectors $\vec{v} = \langle 1, 1, -2\rangle$ and $\vec{w} = \langle 5, -3, 0\rangle$.

2. (a) Find the area of the parallelogram $ABCD$ for the points $A(1, 1, 2)$, $B(4, 0, 2)$, $C(3, 0, 3)$, and $D(0, 1, 3)$.

(b) Find the area of the triangle with vertices $A(2, 5, 4)$, $B(-1, 3, 4)$, and $C(1, 1, -1)$.

(c) Find the volume of the rectangular parallelepiped determined by the vectors $\langle 1, 2, -4\rangle$, $\langle 1, 1, 1\rangle$, and $\langle 0, 3, 1\rangle$.

3. Find the distance from the point $P(1, 0, -3)$ to the line

$$x = 1 + 2t, \quad y = -t, \quad z = 2 + 2t.$$

4. (a) Find the equation of the line which passes through the points $(2, -1, 3)$ and $(4, 2, 1)$.

(b) Where does the line in part (a) intersect the y-z plane?

5. (a) Find the equation of the line which goes through the point $(5, 1, 4)$ and is parallel to the vector $\langle 1, 2, -2\rangle$.

(b) Find the point on the line in part (a) which is closest to the origin.

6. Find the line which passes through the point $(2, -3, 1)$ and is parallel to the line containing the points $(3, 9, 5)$ and $(4, 7, 2)$.

7. Determine whether the following lines are parallel, skew, or intersect. If they intersect, find the point of intersection.

(a)

$$x = 1 + 2t, \quad y = 1 - 4t, \quad z = 5 - t,$$

$$x = 4 - v, \quad y = -1 + 6v, \quad z = 4 + v.$$

(b)

$$x = 3 + t, \quad y = 2 - 4t, \quad z = t,$$

$$x = 4 - s, \quad y = 3 + s, \quad z = -2 + 3s.$$

8. Find the equation of the plane:

(a) Which is perpendicular to the vector $\langle 2, 1, 3\rangle$ and passes through the point $(1, 1, 7)$.

(b) Which contains the point $(4, -5, -1)$ and is perpendicular to the line

$$x = 3 - t, \quad y = 4 + 4t, \quad z = 9.$$

9. Find the point on the plane $2x + y + 3z = 6$ closest to the origin.

10. Find the equation of the plane which goes through the point $(1, 3, -1)$ and is parallel to the plane $3x - 2y + 6z = 9$.

11. Find the equation of the line of intersection of the planes

$$2x - 3y + z = 0 \quad \text{and} \quad x + y + z = 4.$$

12. Find the distance from the point $(-6, 2, 3)$ to the plane $4x - 5y + 8z = 7$.

13. (a) Show that the following lines intersect:

$$x = 2 + 3t, \quad y = -4 - 2t, \quad z = -1 + 4t,$$

$$x = 6 + 4s, \quad y = -2 + 2s, \quad z = -3 - 2s.$$

(b) Find the equation of the plane containing the two lines in part (a).

14. The lines

$$x = 3 + t, \quad y = 2 - 4t, \quad z = t,$$

$$x = 4 - s, \quad y = 3 + s, \quad z = -2 + 3s,$$

are skew. Find the distance between them.

15. Parametrize:

(a) The segment from $P(2, -3, 5)$ to $Q(-10, 0, 6)$.

(b) The curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $2x - 4y + z = 4$.

(c) The curve of intersection of the cylinder $x^2 + 4y^2 = 25$ with the plane $x - 3y + z = 7$.

16. Parametrize:

(a) The surface $x^2 + y^2 + 4z^2 = 9$.

(b) The surface $x^2 + y^2 = 7$,

(c) The surface generated by revolving the curve $y = \sin x$ about the x-axis.

(d) The parallelogram having $P(5, -4, 7)$ as a vertex, where $\vec{v} = \langle 3, 3, 0 \rangle$ and $\vec{w} = \langle 8, -1, 3 \rangle$ are the sides of the parallelogram which emanate from P.

(e) The part of the surface $y = x^2$ lying between the x-y-plane and $z = x + 2y + 3$.

17. The position of a cheesesteak stromboli at time t is

$$\vec{\sigma}(t) = \langle e^{t^2}, \ln(t^2 + 1), \tan t \rangle.$$

(a) Find the velocity and acceleration of the stromboli at $t = 1$.

(b) Find the speed of the stromboli at $t = 1$.

18. The acceleration of a cheeseburger at time t is

$$\vec{a}(t) = \langle 6t, 4, 4e^{2t} \rangle.$$

Find the position $\vec{r}(t)$ if $\vec{v}(1) = \langle 5, 6, 2e^2 - 3\rangle$ and $\vec{r}(0) = \langle 1, 3, 2\rangle$.

19. Compute $\displaystyle\int \left\langle t^2 \cos 2t, t^2 e^{t^3}, \frac{1}{\sqrt{t}}\right\rangle dt$.

20. Find the length of the curve

$$x = \frac{1}{2}t^2 + 1, \quad y = \frac{8}{3}t^{3/2} + 1, \quad z = 8t - 2,$$

from $t = 0$ to $t = 1$.

21. Find the unit tangent vector to:

(a) $\vec{r}(t) = \left\langle t^2 + t + 1, \dfrac{1}{t}, \sqrt{6}t\right\rangle$ at $t = 1$.

(b) $\vec{r}(t) = \langle \cos 5t, \sin 5t, 3t\rangle$.

22. Find the curvature of:

(a) $y = \tan x$ at $x = \dfrac{\pi}{4}$.

(b) $\vec{r}(t) = \langle t^2 + 1, 5t + 1, 1 - t^3\rangle$ at $t = 1$.

Solutions to the Review Sheet for Test 1

1. (a) Find the vector from $P(2, 3, 0)$ to $Q(5, 5, -4)$.

$$\overrightarrow{PQ} = \langle 3, 2, -4\rangle. \quad \square$$

(b) Find two unit vectors perpendicular to the vector $\langle -12, 5\rangle$.

$\langle a, b\rangle$ is perpendicular to $\langle -12, 5\rangle$ if and only if their dot product is 0:

$$\langle -12, 5\rangle \cdot \langle a, b\rangle = 0, \quad -12a + 5b = 0.$$

There are infinitely many pairs of numbers (a, b) which satisfy this equation. For example, $a = 5$ and $b = 12$ works. Thus, $\langle 5, 12\rangle$ is perpendicular to $\langle -12, 5\rangle$.

I can get a unit vector by dividing $\langle 5, 12\rangle$ by its length:

$$\frac{\langle 5, 12\rangle}{\|\langle 5, 12\rangle\|} = \frac{1}{13}\langle 5, 12\rangle.$$

I can get a second unit vector by taking the negative of the first: $-\dfrac{1}{13}\langle 5, 12\rangle$.

Thus, $\pm\dfrac{1}{13}\langle 5, 12\rangle$ are two unit vectors perpendicular to $\langle -12, 5\rangle$. $\square$

(c) Find the vector of length 7 that has the same direction as $\langle 1, 2, -2\rangle$.

Since

$$\|\langle 1, 2, -2\rangle\| = \sqrt{1^2 + 2^2 + (-2)^2} = 3,$$

the vector $\dfrac{1}{3}\langle 1, 2, -2\rangle$ is a unit vector — i.e. a vector with length 1 having the same direction as $\langle 1, 2, -2\rangle$.

Multiplying this vector by 7, I find that $\dfrac{7}{3}\langle 1, 2, -2\rangle$ is a vector of length 7 that has the same direction as $\langle 1, 2, -2\rangle$. $\square$

(d) Find a vector which points in the opposite direction to $\langle 1, -1, 3\rangle$ and has 7 times the length.

Multiplying by -1 gives a vector pointing in the opposite direction; multiplying by 7 gives a vector with 7 times the length. Hence, multiplying by -7 gives a vector which points in the opposite direction and has 7 times the length.

The vector I want is $-7\langle 1, -1, 3\rangle = \langle -7, 7, -21\rangle$. $\square$

(e) Find the scalar component of $\vec{v} = \langle 1, 2, -4\rangle$ in the direction of $\vec{w} = \langle 2, 1, 3\rangle$.

$$\operatorname{comp}_{\vec{w}} \vec{v} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|} = \frac{\langle 1, 2, -4\rangle \cdot \langle 2, 1, 3\rangle}{\|\langle 2, 1, 3\rangle\|} = \frac{2+2-12}{\sqrt{4+1+9}} = -\frac{8}{\sqrt{14}}. \quad \square$$

(f) Find the vector projection of $\vec{v} = \langle 1, 1, -4\rangle$ in the direction of $\vec{w} = \langle 2, 0, 5\rangle$.

$$\operatorname{proj}_{\vec{w}} \vec{v} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w} = \frac{\langle 1, 1, -4\rangle \cdot \langle 2, 0, 5\rangle}{\|\langle 2, 0, 5\rangle\|^2}\langle 2, 0, 5\rangle = \frac{2+0-20}{4+0+25}\langle 2, 0, 5\rangle = -\frac{18}{29}\langle 2, 0, 5\rangle. \quad \square$$

(g) Find the angle in radians between the vectors $\vec{v} = \langle 1, 1, -2\rangle$ and $\vec{w} = \langle 5, -3, 0\rangle$.

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{\langle 1, 1, -2\rangle \cdot \langle 5, -3, 0\rangle}{\|\langle 1, 1, -2\rangle\|\|\langle 5, -3, 0\rangle\|} = \frac{5-3+0}{\sqrt{6}\sqrt{34}} = \frac{2}{\sqrt{204}}, \quad \text{so} \quad \theta = \cos^{-1}\frac{2}{\sqrt{204}} \approx 1.43031. \quad \square$$

2\. (a) Find the area of the parallelogram $ABCD$ for the points $A(1, 1, 2)$, $B(4, 0, 2)$, $C(3, 0, 3)$, and $D(0, 1, 3)$.

Since A, B, C, and D (in that order) are the vertices going around the parallelogram, the vertices adjacent to A are B and D. I have

$$\overrightarrow{AB} = \langle 3, -1, 0\rangle \quad \text{and} \quad \overrightarrow{AD} = \langle -1, 0, 1\rangle.$$

The cross product is

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1, -3, -1\rangle.$$

The area of the parallelogram is the length of the cross product:

$$\|\langle -1, -3, -1\rangle\| = \sqrt{11}. \quad \square$$

(b) Find the area of the triangle with vertices $A(2, 5, 4)$, $B(-1, 3, 4)$, and $C(1, 1, -1)$.

I have

$$\overrightarrow{AB} = \langle -3, -2, 0\rangle \quad \text{and} \quad \overrightarrow{AC} = \langle -1, -4, -5\rangle.$$

The cross product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & -2 & 0 \\ -1 & -4 & -5 \end{vmatrix} = \langle 10, -15, 10\rangle.$$

Since a triangle is *half* of a parallelogram, the area of the triangle is half the length of the cross product:

$$\frac{1}{2}\|\langle 10, -15, 10\rangle\| = \frac{1}{2}\sqrt{425} = \frac{5}{2}\sqrt{17}. \quad \square$$

(c) Find the volume of the rectangular parallelepiped determined by the vectors $\langle 1, 2, -4\rangle$, $\langle 1, 1, 1\rangle$, and $\langle 0, 3, 1\rangle$.

Use the vectors as the rows of a 3×3 matrix and take the determinant:

$$\begin{vmatrix} 1 & 2 & -4 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -16.$$

Therefore, the volume is 16. $\square$

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3. Find the distance from the point $P(1, 0, -3)$ to the line

$$x = 1 + 2t, \quad y = -t, \quad z = 2 + 2t.$$

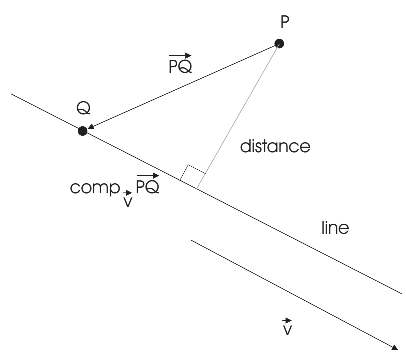
Setting $t = 0$ gives $x = 1$, $y = 0$, and $z = 2$, so the point $Q(1, 0, 2)$ is on the line. Therefore, $\vec{PQ} = \langle 0, 0, 5\rangle$.

The vector $\vec{v} = \langle 2, -1, 2\rangle$ is parallel to the line.

Therefore,

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{\langle 0, 0, 5\rangle \cdot \langle 2, -1, 2\rangle}{\|\langle 2, -1, 2\rangle\|} = \frac{10}{3}.$$

The distance from P to the line is the leg of a right triangle, and I've found the hypotenuse ($|\vec{PQ}|$) and the other leg ($\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}$).



By Pythagoras' theorem,

$$\text{distance} = \sqrt{|\vec{PQ}|^2 - \left(\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}\right)^2} = \sqrt{25 - \frac{100}{9}} = \frac{5\sqrt{5}}{3} \approx 3.72678. \quad \square$$

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4. (a) Find the equation of the line which passes through the points $(2, -1, 3)$ and $(4, 2, 1)$.

The vector from the first point to the second is $\langle 2, 3, -2\rangle$, and this vector is parallel to the line. $(2, -1, 3)$ is a point on the line. Therefore, the line is

$$x - 2 = 2t, \quad y + 1 = 3t, \quad z - 3 = -2t. \quad \square$$

(b) Where does the line in part (a) intersect the y-z plane?

The x-z plane is $x = 0$. Setting $x = 0$ in $x - 2 = 2t$ gives $-2 = 2t$, or $t = -1$. Plugging this into the y and z equations yields $y = -4$ and $z = 5$. Therefore, the line intersects the y-z plane at $(0, -4, 5)$. $\square$

5. (a) Find the equation of the line which goes through the point $(5,1,4)$ and is parallel to the vector $\langle 1,2,-2\rangle$.

$$x-5=t, \quad y-1=2t, \quad z-4=-2t. \quad \square$$

(b) Find the point on the line in part (a) which is closest to the origin.

The distance from the origin $(0,0,0)$ to the point (x,y,z) is

$$\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}=\sqrt{x^2+y^2+z^2}.$$

Since the distance is smallest when its square is smallest, I'll minimize the square of the distance, which is

$$S=x^2+y^2+z^2.$$

(This makes the derivatives easier, since there is no square root.) Since (x,y,z) is on the line, I can substitute $x-5=t$, $y-1=2t$, and $z-4=-2t$ to obtain

$$S=(5+t)^2+(1+2t)^2+(4-2t)^2.$$

Hence,

$$\frac{dS}{dt}=2(5+t)+4(1+2t)-4(4-2t)=-2+18t \quad \text{and} \quad \frac{d^2S}{dt^2}=18.$$

Set $\dfrac{dS}{dt}=0$. I get $-2+18t=0$, or $t=\dfrac{1}{9}$. $\dfrac{d^2S}{dt^2}=18>0$, so the critical point is a local min; since it's the only critical point, it's an absolute min.

Plugging $t=\dfrac{1}{9}$ into the x-y-z equations gives $x=\dfrac{46}{9}$, $y=\dfrac{11}{9}$, and $z=\dfrac{34}{9}$. Thus, the closest point is $\left(\dfrac{46}{9},\dfrac{11}{9},\dfrac{34}{9}\right)$. $\square$

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6. Find the line which passes through the point $(2,-3,1)$ and is parallel to the line containing the points $(3,9,5)$ and $(4,7,2)$.

The vector from $(3,9,5)$ to $(4,7,2)$ is $\langle 1,-2,-3\rangle$; it is parallel to the line containing $(3,9,5)$ and $(4,7,2)$. The line I want to construct is parallel to this line, so it is also parallel to the vector $\langle 1,-2,-3\rangle$.

Since the point $(2,-3,1)$ is on the line I want to construct, the line is

$$x-2=t, \quad y+3=-2t, \quad z-1=-3t. \quad \square$$

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7. Determine whether the following lines are parallel, skew, or intersect. If they intersect, find the point of intersection.

(a)

$$x=1+2t, \quad y=1-4t, \quad z=5-t,$$

$$x=4-v, \quad y=-1+6v, \quad z=4+v.$$

The vector $\langle 2,-4,-1\rangle$ is parallel to the first line. The vector $\langle -1,6,1\rangle$ is parallel to the second line. The vectors aren't multiples of one another, so the vectors aren't parallel. Therefore, the lines aren't parallel.

Next, I'll check whether the lines intersect.

Solve the x-equations simultaneously:

$$1+2t=4-v, \quad v=3-2t.$$

Set the y-expressions equal, then plug in $v = 3 - 2t$ and solve for t:

$$1 - 4t = -1 + 6v, \quad 1 - 4t = -1 + 6(3 - 2t), \quad 1 - 4t = 17 - 12t, \quad t = 2.$$

Therefore, $v = 3 - 2t = -1$.

Check the values for consistency by plugging them into the z-equations:

$$z = 5 - t = 5 - 2 = 3, \quad z = 4 + v = 4 + (-1) = 3.$$

The equations are consistent, so the lines intersect. If I plug $t = 2$ into the x-y-z equations, I obtain $x = 5$, $y = -7$, and $z = 3$. The lines intersect at $(5, -7, 3)$. $\square$

(b)

$$x = 3 + t, \quad y = 2 - 4t, \quad z = t,$$
$$x = 4 - s, \quad y = 3 + s, \quad z = -2 + 3s.$$

The vector $\langle 1, -4, 1\rangle$ is parallel to the first line. The vector $\langle -1, 1, 3\rangle$ is parallel to the second line. The vectors aren't multiples of one another, so the vectors aren't parallel. Therefore, the lines aren't parallel.

Next, I'll check whether the lines intersect.

Solve the x-equations simultaneously:

$$3 + t = 4 - s, \quad t = 1 - s.$$

Set the y-expressions equal, then plug in $t = 1 - s$ and solve for s:

$$2 - 4t = 3 + s, \quad 2 - 4(1 - s) = 3 + s, \quad -2 + 4s = 3 + s, \quad s = \frac{5}{3}.$$

Therefore, $t = 1 - s = -\frac{2}{3}$.

Check the values for consistency by plugging them into the z-equations:

$$z = t = -\frac{2}{3}, \quad z = -2 + 3s = 3.$$

The equations are inconsistent, so the lines do not intersect.

Since the lines aren't parallel and don't intersect, they must be skew. $\square$

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8. Find the equation of the plane:

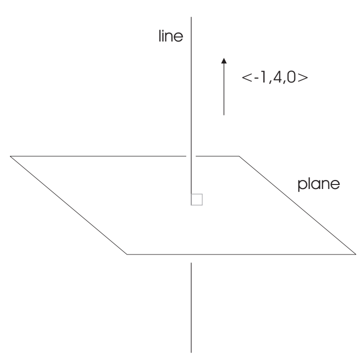
(a) Which is perpendicular to the vector $\langle 2, 1, 3\rangle$ and passes through the point $(1, 1, 7)$.

$$2(x - 1) + (y - 1) + 3(z - 7) = 0, \quad \text{or} \quad 2x + y + 3z = 24. \quad \square$$

(b) Which contains the point $(4, -5, -1)$ and is perpendicular to the line

$$x = 3 - t, \quad y = 4 + 4t, \quad z = 9.$$

The vector $\langle -1, 4, 0\rangle$ is parallel to the line. The line is perpendicular to the plane. Hence, the vector $\langle -1, 4, 0\rangle$ is perpendicular to the plane.



The point $(4, -5, -1)$ is on the plane.
Therefore, the plane is

$$-(x-4)+4(y+5)+(0)(z+1)=0, \quad \text{or} \quad -x+4y+24=0. \quad \square$$

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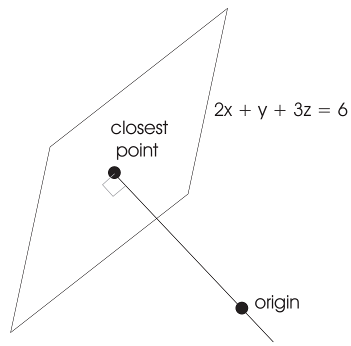
9. Find the point on the plane $2x+y+3z=6$ closest to the origin.

I'll do this by constructing a line which is perpendicular to the plane and passes through the origin. The desired point is the point where the line intersects the plane.



The vector $\langle 2, 1, 3\rangle$ is perpendicular to the plane. Hence, the line

$$x=2t, \quad y=t, \quad z=3t$$

passes through the origin and is perpendicular to the plane.

Find the point where the line intersects the plane by substituting these expressions into the plane equation and solving for t:

$$2\cdot 2t+t+3\cdot 3t=6, \quad 14t=6, \quad t=\frac{3}{7}.$$

Finally, plugging this back into the x-y-z equations yields $x=\dfrac{6}{7}$, $y=\dfrac{3}{7}$, and $z=\dfrac{9}{7}$. Hence, the point on the plane closest to the origin is $\left(\dfrac{6}{7}, \dfrac{3}{7}, \dfrac{9}{7}\right)$. $\square$

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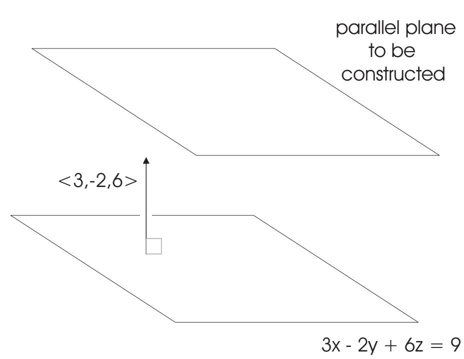
10. Find the equation of the plane which goes through the point $(1, 3, -1)$ and is parallel to the plane $3x-2y+6z=9$.

The vector $\langle 3, -2, 6\rangle$ is perpendicular to $3x-2y+6z=9$.

Since the plane I want to construct is parallel to the given plane, the vector $\langle 3, -2, 6\rangle$ is perpendicular to the plane I want to construct.



The plane I want to construct contains the point $(1, 3, -1)$.
Hence, the plane is

$$3(x-1) - 2(y-3) + 6(z+1) = 0, \quad \text{or} \quad 3x - 2y + 6z = -6. \quad \square$$

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11. Find the equation of the line of intersection of the planes

$$2x - 3y + z = 0 \quad \text{and} \quad x + y + z = 4.$$

Set $z = t$, then multiply the second equation by 3:

$$\begin{array}{lcl} 2x - 3y + t = 0 & \rightarrow & 2x - 3y + t = 0 \\ x + y + t = 4 & \rightarrow & 3x + 3y + 3t = 12 \end{array}$$

Add the equations and solve for x:

$$5x + 4t = 12, \quad x = \frac{12}{5} - \frac{4}{5}t.$$

Plug this back tino $x + y + t = 4$ and solve for y:

$$\left(\frac{12}{5} - \frac{4}{5}t\right) + y + t = 4, \quad y = \frac{8}{5} - \frac{1}{5}t.$$

Therefore, the line of intersection is

$$x = \frac{12}{5} - \frac{4}{5}t, \quad y = \frac{8}{5} - \frac{1}{5}t, \quad z = t.$$

Note: You can also do this by taking vectors perpendicular to the two planes and computing their cross product. This gives a vector parallel to the line of intersection. Find a point on the line of intersection by setting $z = 0$ (say) and solving the plane equations simultaneously. Then plug the point and the cross product vector into the parametric equations for the line. $\square$

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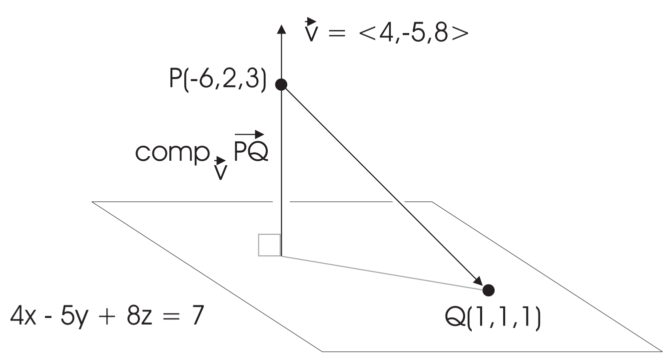
12. Find the distance from the point $(-6, 2, 3)$ to the plane $4x - 5y + 8z = 7$.

The vector $\vec{v} = \langle 4, -5, 8 \rangle$ is perpendicular to the plane.

To find a point on the plane, set y and z equal to numbers (chosen at random) and solve for x. For example, set $y = 1$ and $z = 1$. Then

$$4x - 5 + 8 = 7, \quad \text{so} \quad 4x = 4, \quad \text{or} \quad x = 1.$$

Thus, the point $Q(1, 1, 1)$ is on the plane. The vector from $P(-6, 2, 3)$ to Q is $\langle 7, -1, -2 \rangle$.



The distance is the absolute value of the component of $\overrightarrow{PQ}$ in the direction of $\vec{v}$:

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{\langle 7, -1, -2\rangle \cdot \langle 4, -5, 8\rangle}{\|\langle 4, -5, 8\rangle\|} = \frac{17}{\sqrt{105}} \approx 1.65703. \quad \square$$

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13. (a) Show that the following lines intersect:

$$x = 2 + 3t, \quad y = -4 - 2t, \quad z = -1 + 4t,$$

$$x = 6 + 4s, \quad y = -2 + 2s, \quad z = -3 - 2s.$$

Solve the x-equations simultaneously:

$$2 + 3t = 6 + 4s, \quad t = \frac{4}{3} + \frac{4}{3}s.$$

Set the y-expressions equal, then plug in $t = \frac{4}{3} + \frac{4}{3}s$ and solve for s:

$$-4 - 2t = -2 + 2s, \quad -4 - 2\left(\frac{4}{3} + \frac{4}{3}s\right) = -2 + 2s, \quad -\frac{20}{3} - \frac{8}{3}s = -2 + 2s, \quad s = -1.$$

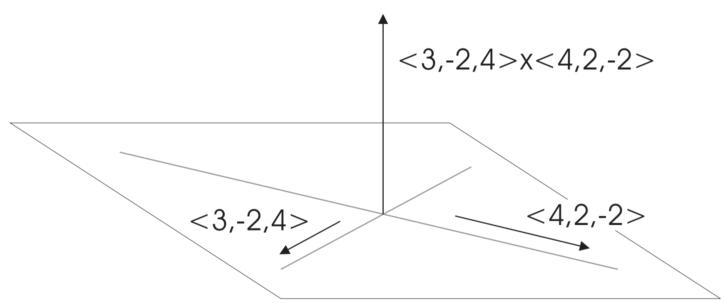
Therefore, $t = \frac{4}{3} + \frac{4}{3}s = 0$.
Check the values for consistency by plugging them into the z-equations:

$$z = -1 + 4t = -1, \quad z = -3 - 2s = -1.$$

The equations are consistent, so the lines intersect. If I plug $t = 0$ into the x-y-z equations, I obtain $x = 2$, $y = -4$, and $z = -1$. The lines intersect at $(2, -4, -1)$. $\square$

(b) Find the equation of the plane containing the two lines in part (a).

The vector $\langle 3, -2, 4\rangle$ is parallel to the first line, and the vector $\langle 4, 2, -2\rangle$ is parallel to the second line. The cross product of the vectors is perpendicular to the plane containing the lines.



The cross product is

$$\langle 3, -2, 4\rangle \times \langle 4, 2, -2\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 4 \\ 4 & 2 & -2 \end{vmatrix} = \langle -4, 22, 14\rangle.$$

From part (a), the point $(2, -4, -1)$ is on both lines, so it is surely in the plane. Therefore, the plane is

$$-4(x - 2) + 22(y + 4) + 14(z + 1) = 0, \quad \text{or} \quad -4x + 22y + 14z = -110, \quad \text{or} \quad 2x - 11y - 7z = 55. \quad \square$$

14. The lines

$$x = 3 + t, \quad y = 2 - 4t, \quad z = t,$$

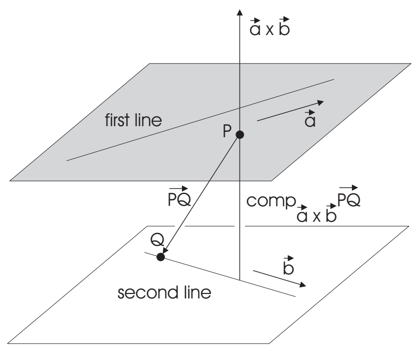
$$x = 4 - s, \quad y = 3 + s, \quad z = -2 + 3s,$$

are skew. Find the distance between them.

I'll find vectors $\vec{a}$ and $\vec{b}$ parallel to the lines and take their cross product. This gives a vector perpendicular to both lines. (If you think of the skew lines as lying in the ceiling and the floor of a room, the cross product vector $\vec{a} \times \vec{b}$ will be perpendicular to both the ceiling and the floor.)

Next,I'll find a point P on the first line and a point Q on the second line.

Finally, the distance between the lines — in my analogy, the distance between the ceiling and the floor — will be $|\operatorname{comp}_{\vec{a}\times\vec{b}} \overrightarrow{PQ}|$.



The vector $\vec{a} = \langle 1, -4, 1 \rangle$ is parallel to the first line, and the vector $\vec{b} = \langle -1, 1, 3 \rangle$ is parallel to the second line. Hence, their cross product is

$$\langle 1, -4, 1 \rangle \times \langle -1, 1, 3 \rangle = \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -4 & 1 \\ -1 & 1 & 3 \end{array} \right| = \langle -13, -4, -3 \rangle.$$

Setting $t = 0$ in the first line gives $x = 3$, $y = 2$, and $z = 0$. Hence, the point $P(3, 2, 0)$ lies in the first line. Setting $s = 0$ in the second line gives $x = 4$, $y = 3$, and $z = -2$, Hence, the point $Q(4, 3, -2)$ lies in the second line. The vector from one point to the other is $\overrightarrow{PQ} = \langle 1, 1, -2 \rangle$.

Next,

$$\operatorname{comp}_{\vec{a}\times\vec{b}} \overrightarrow{PQ} = \frac{\langle 1, 1, -2 \rangle \cdot \langle -13, -4, -3 \rangle}{\|\langle -13, -4, -3 \rangle\|} = -\frac{11}{\sqrt{194}}.$$

Therefore, the distance is

$$\text{distance} = \frac{11}{\sqrt{194}} \approx 0.78975. \quad \square$$

15. Parametrize:

(a) The segment from $P(2, -3, 5)$ to $Q(-10, 0, 6)$.

$$(x, y, z) = (1 - t)(2, -3, 5) + t(-10, 0, 6) = (2 - 12t, -3 + 3t, 5 + t),$$

or

$$x = 2 - 12t, \quad y = -3 + 3t, \quad z = 5 + t \quad \text{for} \quad 0 \le t \le 1. \quad \square$$

(b) The curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $2x - 4y + z = 4$.

Solving the plane equation for z gives $z = 4 - 2x + 4y$. Equate the two expressions for z, then complete the square in x and y:

$$x^2 + y^2 = 4 - 2x + 4y, \quad x^2 + 2x + y^2 - 4y = 4, \quad x^2 + 2x + 1 + y^2 - 4y + 4 = 9, \quad (x+1)^2 + (y-2)^2 = 9.$$

This equation represents the projection of the curve of intersection into the x-y plane. It's a circle with center $(-1, 2)$ and radius 3. It may be parametrized by

$$x = 3(\cos t - 1), \quad y = 3(\sin t + 2).$$

Plug this back into either z-equation to find z. I'll use the plane equation:

$$z = 4 - 2\left(3(\cos t - 1)\right) + 4\left(3(\sin t + 2)\right) = 34 - 6\cos t + 12\sin t.$$

The curve of intersection is

$$x = 3(\cos t - 1), \quad y = 3(\sin t + 2), \quad z = 34 - 6\cos t + 12\sin t. \quad \square$$

(c) The curve of intersection of the cylinder $x^2 + 4y^2 = 25$ with the plane $x - 3y + z = 7$.

The *curve* $x^2 + 4y^2 = 25$ may be parametrized by

$$x = 5\cos t, \quad y = \frac{5}{2}\sin t.$$

Solving the plane equation for z yields $z = 7 - x + 3y$. Plug in the expressions for x and y:

$$z = 7 - 5\cos t + \frac{15}{2}\sin t.$$

The curve of intersection is

$$x = 5\cos t, \quad y = \frac{5}{2}\sin t, \quad z = 7 - 5\cos t + \frac{15}{2}\sin t. \quad \square$$

---

16. Parametrize:

(a) The surface $x^2 + y^2 + 4z^2 = 9$.

$$x = 3\cos u\cos v, \quad y = 3\sin u\cos v, \quad z = \frac{3}{2}\sin v. \quad \square$$

(b) The surface $x^2 + y^2 = 7$,

$$x = \sqrt{7}\cos u, \quad y = \sqrt{7}\sin u, \quad z = v. \quad \square$$

(c) The surface generated by revolving the curve $y = \sin x$ about the x-axis.

The curve $y = \sin x$ may be parametrized by

$$x = u, \quad y = \sin u.$$

Therefore, the surface generated by revolving the curve $y = \sin x$ about the x-axis may be parametrized by

$$x = u, \quad y = \sin u\cos v, \quad z = \sin u\sin v. \quad \square$$

(d) The parallelogram having $P(5,-4,7)$ as a vertex, where $\vec{v} = \langle 3,3,0\rangle$ and $\vec{w} = \langle 8,-1,3\rangle$ are the sides of the parallelogram which emanate from P.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 3 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \\ 7 \end{bmatrix}, \quad \text{or} \quad x = 3u + 8v + 5, \quad y = 3u - v - 4, \quad z = 3v + 7,$$

where $0 \le u \le 1$ and $0 \le v \le 1$. $\square$

(e) The part of the surface $y = x^2$ lying between the x-y-plane and $z = x + 2y + 3$.

I'll use a segment parametrization. The curve $y = x^2$ may be parametrized by

$$x = u, \quad y = u^2.$$

Thus, a typical point on the intersection of the surface $y = x^2$ with the x-y-plane has coordinates $(u, u^2, 0)$, since the x-y-plane is $z = 0$.

Plugging $x = u$ and $y = u^2$ into $z = x + 2y + 3$ yields $z = u + 2u^2 + 3$. Therefore, the point on the plane $z = x + 2y + 3$ which lies directly above $(u, u^2, 0)$ is $(u, u^2, u + 2u^2 + 3)$.

The segment joining $(u, u^2, 0)$ to $(u, u^2, u + 2u^2 + 3)$ is

$$(x, y, z) = (1 - v)(u, u^2, 0) + v(u, u^2, u + 2u^2 + 3) = (u, u^2, 3v + uv + 2u^2 v).$$

Therefore, the surface may be parametrized by

$$x = u, \quad y = u^2, \quad z = 3v + uv + 3u^2 v,$$

where $0 \le v \le 1$. $\square$

---

17. The position of a cheesesteak stromboli at time t is

$$\vec{\sigma}(t) = \langle e^{t^2}, \ln(t^2 + 1), \tan t\rangle.$$

(a) Find the velocity and acceleration of the stromboli at $t = 1$.

$$\vec{v}(t) = \vec{\sigma}\,'(t) = \left\langle 2te^{t^2}, \frac{2t}{t^2 + 1}, (\sec t)^2 \right\rangle,$$

$$\vec{a}(t) = \vec{v}\,'(t) = \left\langle 4t^2 e^{t^2} + 2e^{t^2}, \frac{(t^2 + 1)(2) - (2t)(2t)}{(t^2 + 1)^2}, 2(\sec t)^2 \tan t \right\rangle.$$

Hence,

$$\vec{v}(1) = \langle 2e, 1, (\sec 1)^2\rangle \quad \text{and} \quad \vec{a}(1) = \left\langle 6e, 0, 2(\sec 1)^2 \tan 1 \right\rangle. \quad \square$$

(b) Find the speed of the stromboli at $t = 1$.

$$\|\vec{v}(1)\| = \sqrt{(2e)^2 + 1^2 + ((\sec 1)^2)^2} = \sqrt{4e^2 + 1 + (\sec 1)^4}. \quad \square$$

---

18. The acceleration of a cheeseburger at time t is

$$\vec{a}(t) = \langle 6t, 4, 4e^{2t}\rangle.$$

Find the position $\vec{r}(t)$ if $\vec{v}(1) = \langle 5, 6, 2e^2 - 3\rangle$ and $\vec{r}(0) = \langle 1, 3, 2\rangle$.

Since the acceleration function is the derivative of the velocity function, the velocity function is the integral of the acceleration function:

$$\vec{v}(t) = \int \langle 6t, 4, 4e^{2t} \rangle \, dt = \left\langle 3t^2, 4t, 2e^{2t} \right\rangle + \langle c_1, c_2, c_3 \rangle.$$

In order to find the arbitrary constant vector $\langle c_1, c_2, c_3 \rangle$, I'll plug the initial condition $\vec{v}(1) = \langle 5, 6, 2e^2 - 3 \rangle$ into the equation for $\vec{v}(t)$:

$$\langle 5, 6, 2e^2 - 3 \rangle = \vec{v}(1) = \langle 3, 4, 2e^2 \rangle + \langle c_1, c_2, c_3 \rangle, \quad \langle 2, 2, -3 \rangle = \langle c_1, c_2, c_3 \rangle.$$

Therefore,

$$\vec{v}(t) = \left\langle 3t^2, 4t, 2e^{2t} \right\rangle + \langle 2, 2, -3 \rangle = \left\langle 3t^2 + 2, 4t + 2, 2e^{2t} - 3 \right\rangle.$$

Since the velocity function is the derivative of the position function, the position function is the integral of the velocity function:

$$\vec{r}(t) = \int \left\langle 3t^2 + 2, 4t + 2, 2e^{2t} - 3 \right\rangle \, dt = \left\langle t^3 + 2t, 2t^2 + 2t, e^{2t} - 3t \right\rangle + \langle d_1, d_2, d_3 \rangle.$$

In order to find the arbitrary constant vector $\langle d_1, d_2, d_3 \rangle$, I'll plug the initial condition $\vec{r}(0) = \langle 1, 3, 2 \rangle$ into the equation for $\vec{r}(t)$:

$$\langle 1, 3, 2 \rangle = \vec{r}(0) = \langle 0, 0, 1 \rangle + \langle d_1, d_2, d_3 \rangle, \quad \langle 1, 3, 1 \rangle = \langle d_1, d_2, d_3 \rangle.$$

Therefore,

$$\vec{r}(t) = \left\langle t^3 + 2t, 2t^2 + 2t, e^{2t} - 3t \right\rangle + \langle 1, 3, 1 \rangle = \left\langle t^3 + 2t + 1, 2t^2 + 2t + 3, e^{2t} - 3t + 1 \right\rangle. \quad \square$$

---

19. Compute $\displaystyle \int \left\langle t^2 \cos 2t, t^2 e^{t^3}, \frac{1}{\sqrt{t}} \right\rangle dt$.

I'll compute the integral of each component separately.

$$\begin{array}{ccccc} & \dfrac{d}{dt} & & \displaystyle\int dt \\ + & t^2 & & \cos 2t \\ & & \searrow & \\ - & 2t & & \dfrac{1}{2}\sin 2t \\ & & \searrow & \\ + & 2 & & -\dfrac{1}{4}\cos 2t \\ & & \searrow & \\ - & 0 & & -\dfrac{1}{8}\sin 2t \end{array}$$

$$\int t^2 \cos 2t \, dt = \frac{1}{2} t^2 \sin 2t + \frac{1}{2} t \cos 2t - \frac{1}{4} \sin 2t + C.$$

$$\int t^2 e^{t^3} \, dt = \int t^2 e^u \cdot \frac{du}{3t^2} = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{t^3} + C.$$

$$\left[u = t^3, \quad du = 3t^2 \, dt, \quad dt = \frac{du}{3t^2} \right]$$

$$\int \frac{1}{\sqrt{t}}\, dt = 2\sqrt{t} + C.$$

Therefore,

$$\int \left\langle t^2 \cos 2t, t^2 e^{t^3}, \frac{1}{\sqrt{t}} \right\rangle dt = \left\langle \frac{1}{2}t^2 \sin 2t + \frac{1}{2}t\cos 2t - \frac{1}{4}\sin 2t, \frac{1}{3}e^{t^3}, 2\sqrt{t} \right\rangle + \vec{c}. \quad \square$$

---

20. Find the length of the curve

$$x = \frac{1}{2}t^2 + 1, \quad y = \frac{8}{3}t^{3/2} + 1, \quad z = 8t - 2,$$

from $t = 0$ to $t = 1$.

$$\frac{dx}{dt} = t, \quad \frac{dy}{dt} = 4t^{1/2}, \quad \frac{dz}{dt} = 8,$$

$$\left(\frac{dx}{dt}\right)^2 = t^2, \quad \left(\frac{dy}{dt}\right)^2 = 16t, \quad \left(\frac{dz}{dt}\right)^2 = 64,$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{t^2 + 16t + 64} = \sqrt{(t+8)^2} = t + 8.$$

The length of the curve is

$$L = \int_0^1 (t+8)\, dt = \left[\frac{1}{2}t^2 + 8t\right]_0^1 = \frac{17}{2}. \quad \square$$

---

21. Find the unit tangent vector to:

(a) $\vec{r}(t) = \left\langle t^2 + t + 1, \frac{1}{t}, \sqrt{6}t \right\rangle$ at $t = 1$.

$$\vec{r}\,'(t) = \left\langle 2t + 1, -\frac{1}{t^2}, \sqrt{6} \right\rangle, \quad \text{so} \quad \vec{r}\,'(1) = \langle 3, -1, \sqrt{6} \rangle.$$

Since $\|\vec{r}\,'(1)\| = \sqrt{9 + 1 + 6} = 4$, the unit tangent vector is

$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \frac{1}{4}\langle 3, -1, \sqrt{6} \rangle. \quad \square$$

(b) $\vec{r}(t) = \langle \cos 5t, \sin 5t, 3t \rangle$.

$$\vec{r}\,'(t) = \langle -5\sin 5t, 5\cos 5t, 3 \rangle,$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-5\sin 5t)^2 + (5\cos 5t)^2 + 3^2} = \sqrt{25(\sin 5t)^2 + 25(\cos 5t)^2 + 9} = \sqrt{25 + 9} = \sqrt{34}.$$

Hence, the unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{34}}\langle -5\sin 5t, 5\cos 5t, 3 \rangle. \quad \square$$

---

22. Find the curvature of:

(a) $y = \tan x$ at $x = \dfrac{\pi}{4}$.

Since $y = \tan x$ is a curve in the x-y-plane, I'll use the formula

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

First,

$$f'(x) = (\sec x)^2 \quad \text{and} \quad f''(x) = 2(\sec x)^2 \tan x.$$

Plug in $x = \dfrac{\pi}{4}$:

$$f'\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2, \quad f''\left(\frac{\pi}{4}\right) = 2(\sqrt{2})^2(1) = 4.$$

The curvature is

$$\kappa = \frac{|4|}{(1 + 2^2)^{3/2}} = \frac{4}{5\sqrt{5}}. \quad \square$$

(b) $\vec{r}(t) = \langle t^2 + 1, 5t + 1, 1 - t^3 \rangle$ at $t = 1$.

In this case, I'll use the formula

$$\kappa = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}.$$

First,

$$\vec{r}\,'(t) = \langle 2t, 5, 3t^2 \rangle \quad \text{and} \quad \vec{r}\,''(t) = \langle 2, 0, 6t \rangle.$$

Hence,

$$\vec{r}\,'(1) = \langle 2, 5, 3 \rangle \quad \text{and} \quad \vec{r}\,''(1) = \langle 2, 0, 6 \rangle.$$

Next,

$$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 5 & 3 \\ 2 & 0 & 6 \end{vmatrix} = \langle 30, -6, -10 \rangle.$$

The curvature is

$$\kappa = \frac{\|\langle 30, -6, -10 \rangle\|}{\|\langle 2, 5, 3 \rangle\|^3} = \frac{\sqrt{900 + 36 + 100}}{(\sqrt{4 + 25 + 9})^3} = \frac{\sqrt{1036}}{38\sqrt{38}}. \quad \square$$

Real generosity toward the future lies in giving all to the present. - Albert Camus